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ON WEAKLY PRECIPITOUS FILTERS

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ABSTRACT

We answer a question of T. Jech, showing that (1) there may exist weakly precipitous filters in L, and (2) there may exist a weakly precipitous filter on ω_1 in a set-generic extension of L. Hence, the existence of a weakly precipitous filter on ω_1 does not imply the existence of $0^{\#}$.

1. Introduction

1.1. After [Si], [GH] established some bounds on the power of singular cardinals. A typical application of [GH] is the following.

(A) Assume that $\aleph_{\omega_1}$ is a strong limit cardinal. Then

$$2^{\aleph_{\omega_1}} < \aleph((2^{\omega_1})^+).$$

1.2. Let $N: On \to Cn = \{ = \alpha \mid \alpha$ is an infinite cardinal$\}$ be a normal functional, and set $C = \mathrm{Rg}(N)$. We shall let N' denote the normal enumeration of the class C' of all the fixed points of N.

1.3. The following question was left open in [GH].

(B) Assume that $\aleph'_{\omega_1}$ is a strong limit cardinal. Is there an "expressible" bound on $2^{\aleph'_{\omega_1}}$, like for example $2^{\aleph'_{\omega_1}} < \aleph'((2^{\omega_1})^+)$.

1.4. In [JP], the answer to question (B) was shown to be positive if there exists a precipitous filter on ω_1.

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1.5. The previous assumption has the consistency strength of the existence of a measurable cardinal. Hence, the answer of [JP] is not done in ZFC.

1.6. [S] gets rid of the supplementary assumption, introducing a new kind of filter, and shows the following:

(C) If $\aleph'_{\omega_1}$ is strong limit cardinal and there exists over ω_1 a weakly precipitous filter, then $2^{\aleph'_{\omega_1}} < \aleph'((2^{2^{\omega_1}})^+)$.

(D) If $\aleph'_{\omega_1}$ is a strong limit and $2^{\aleph'_{\omega_1}} > (\aleph'_{\omega_1})^+$, then there exists a weakly precipitous filter on ω_1.

For (C), see [S, Theorem 6.6] and for (D), [S, Conclusion 4.14 and Theorem 4.15]. (D) is proven with the help of the covering lemma and of the following:

(E) If there exists a cardinal λ such that $\lambda \to ((2^{2^{\omega_1}})^+)_2^{<\omega}$, then there exists a weakly precipitous filter on ω_1.

1.7. The filters in question are called "almost nice" in [S] and "weakly precipitous" in [J].

Since a natural way to obtain such a filter is (E) above, a natural question is the following, asked in [J]:

(F) Does the existence of a weakly precipitous filter on ω_1 imply the existence of $0^\#$?

1.8. We shall show that the answer to question (F) is negative. Our sequence of results is as follows.

(1) We define semi-precipitous filters. Every semi-precipitous filter is weakly precipitous.

Let us say that κ is semi-precipitous (resp.: weakly precipitous) iff it bears a semi-precipitous (resp. a weakly precipitous) filter. Then:

(2) If κ is semi-precipitous, then $L \models$ "κ is semi-precipitous".

(3) Assume that $0^\#$ exists. Then: every Silver indiscernible is semi-precipitous in L.

On the other hand, $\mathrm{ZFC} + V = L \models$ "if κ is semi-precipitous, then κ is inaccessible, indeed completely ineffable". Hence, in order to obtain a semi-precipitous filter on ω_1, we show

(4) If κ is semi-precipitous and P is a set of conditions having the κ-antichain condition, then $V^P \models$ "κ is semi-precipitous".

We also observe that "weakly precipitous" is, consistencywise, strictly weaker that "semi-precipitous", by showing

(5) ZFC + $V = L \models$ "if κ is semi-precipitous, then, for some $\alpha < \kappa$, α is weakly precipitous", and quoting that κ must be inaccessible.

Hence, we prove finally

(6) Assume that κ is weakly precipitous and that P is a set of conditions which satisfies the κ-antichain condition. Then: $V^P \models$ "κ is weakly precipitous".

2. Notations

2.1. On denotes the class of all ordinals. For $a \subseteq On$, ot(a) is the order-type of a.

2.2. $|X|$ is the cardinality of X. cof(λ) is the cofinality of $\lambda \in On$.

2.3. If λ is a cardinal, H_λ is the set of all sets of hereditary cardinality strictly less that λ.

2.4. If j is an elementary embedding, cp(j) is the critical point of j.

2.5. If p is a function, dom(p) is the domain of p and Rg(p) the range of p.

2.6. If P is a set of conditions, $B(P)$ is the boolean completion of P. If G is P-generic over V and $a \in V^P$, a_G is the G-interpretation of a in $V[G]$.

2.7. Coll(ω, θ) is the set of all finite partial functions $p : \omega \to \theta$, ordered by reverse inclusion.

2.8. If $\boldsymbol{A}, \boldsymbol{B}$ are structures, $\boldsymbol{A} \prec \boldsymbol{B}$ means that $\boldsymbol{A}$ is an elementary substructure of $\boldsymbol{B}$.

2.9. Let X be a set, and let F be a filter over X. We set $I_F = \{S \subseteq X / X - S \in F\}$ (the ideal dual to F),

$$F^+ = \{S \subseteq X / S \notin I_F\} = \{S \subseteq X / \forall E \in F, S \cap E \neq \varnothing\}.$$

We also set $B(F) = \mathscr{P}(X)/I_F$ (the boolean algebra of F). For $S \subseteq X$, $[S]_F$

denotes its class in $B(F)$. We set $[S]_F \leqq [T]_F$ iff there exists some $E \in F$ such that $S \cap E \subseteq T$.

Note that $B(F) - \{0\}$ is the separative ordered set associated with the (non-separative) ordered set $(F^+, \subseteq)$.

For $A \in F^+$, we denote by $F[A]$ the filter generated on X by $F \cup \{A\}$. Hence, $F[A] = \{S \subseteq X \mid \exists E \in F, E \cap A \subseteq S\}$.

2.10. Finally, if F is a filter on X and $f, g \in On^X$, we set: $f <_F g$ iff $\{x \in X \mid f(x) < g(x)\} \in F$, $f \leqq_F g$ iff $\{x \in X \mid f(x) \leqq g(x)\} \in F$ and $f =_F g$ iff $(f \leqq_F g$ and $g \leqq_F f)$.

3. Weakly precipitous filters

3.1. κ will always denote an uncountable regular cardinal. All the filters over κ will be assumed to be normal. For the convenience of the reader, we recall some definitions and facts from [S].

3.2. *The game* $G(F, g, \alpha)$. Let F be a normal filter over κ. Let $g \in On^\kappa$ and $\alpha \in On$. We consider the following game, denoted by $G(F, g, \alpha)$, of (potential) length ω. Set $F_0 = F$, $g_0 = g$, $\alpha_0 = \alpha$. For $1 \leqq i < \omega$, the move number i of player I will be a pair (A_i, g_i), with $A_i \subseteq \kappa$ and $g_i \in On^\kappa$, while the move of player II will be a pair (F_i, α_i), where F_i is a normal filter on κ and $\alpha_i \in On$. The rules are as follows.

(a) For $0 \leqq i < \omega$, $A_{i+1} \in (F_i)^+$ and $g_{i+1} <_{F_i[A_{i+1}]} g_i$.

(b) For $0 \leqq i < \omega$, $F_{i+1} \supseteq F_i[A_{i+1}]$ and $\alpha_{i+1} < \alpha_i$.

It is clear that, at some stage $i < \omega$, one of the two players is not going to be able to play according to the rules. The first player to whom this happens has lost the game. Hence, $G(F, g, \alpha)$ is an open and closed game — in particular it is determined.

3.3. We shall shorten the expression "player X has a winning strategy in the game G" to "X wins G".

3.4. Remarks.

(1) If II wins $G(F, g, \alpha)$ and $\alpha \leqq \beta$, II wins $G(F, g, \beta)$.

(2) If II wins $G(F, g, \alpha)$ and $g' \leqq_F g$, then II wins $G(F, g', \alpha)$.

(3) If II wins $G(F, g, On)$ then, for some $\delta \in On$, II wins $G(F, g, \delta)$.

Indeed, setting $\theta = 2^\kappa \cdot \Pi_{\alpha<\kappa} | g(\alpha) + 1 |$, we see that player I has essentially θ partial plays in $G(F, g, On)$, since we can always assume that all moves (A_i, g_i) of I are such that $g_i \in \Pi_{\alpha<\kappa}(g(\alpha) + 1)$. Hence, a winning strategy for II will yield at most θ possible answers. Consequently, if II wins $G(F, g, On)$, then, for some $\delta < \theta^+$, II wins $G(F, g, \delta)$.

3.5. *The game* $G^*(F, g)$. We still let F be a normal filter over κ and $g \in On^\kappa$. $G^*(F, g)$ is a game of length ω. It runs like $G(F, g, \alpha)$, but "forgetting" the α_i's. I.e.: for $1 \leqq i < \omega$, player I plays $(A_i, g_i) \in \mathscr{P}(\kappa) \times On^\kappa$ and player II plays a normal filter F_i on κ. The rules are as rules (a), (b) of §3.2, with "$\alpha_{i+1} < \alpha_i$" omitted. I wins $G^*(F, g)$ iff he (or she) can play ω correct moves. If not, then II wins. Hence, $G^*(F, g)$ is a closed game.

3.6. REMARKS.
(1) It is clear that, if II wins $G(F, g, On)$, then II wins $G^*(F, g)$.
(2) Conversely, if II wins $G^*(F, g)$, then II wins $G(F, g, On)$.

To see that, let σ^* be a winning strategy for II in $G^*(F, g)$. Let T be the set of all finite sequences $s = ((A_1, g_1), \ldots, (A_n, g_n))$, where $A_i \subseteq \kappa$, $g_i \in \Pi_{\alpha<\kappa}(g(\alpha) + 1)$ and s is a correct partial play of I against σ^* in $G^*(F, g)$. Let us order T by end-extension, setting $s \lhd t$ iff $s \subseteq t$ and $s \neq t$. Since σ^* is a winning strategy, $\lhd$ is well-founded on T. Hence, $(T, \lhd)$ admits a (minimal) rank function $\rho : T \to On$. Set $\theta = 2^\kappa \cdot \Pi_{\alpha<\kappa} |g(\alpha) + 1|$. Since $|T| \leqq \theta$, there exists $\delta < \theta^+$ such that $\mathrm{Rg}(\rho) \subseteq \delta$. It is now clear that II has a winning strategy, say σ, in the game $G(F, g, \delta)$: if I has played $s = ((A_1, g_1), \ldots, (A_n, g_n))$, then II plays $\sigma(s) = (F_n, \alpha_n)$, where $F_n = \sigma^*(s)$ and $\alpha_n = \rho(s)$.

(3) Assume that II wins $G^*(F, (2^{2^\kappa})^+)$ [where we denote by the same letter the ordinal λ and the constant function $\kappa \to \{\lambda\}$]. Then, for all $g \in On^\kappa$, II wins $G^*(F, g)$.

To see this, assume that II does not win $G^*(F, g)$. Hence, I wins $G^*(F, g)$. Let τ be a winning strategy for I in $G^*(F, g)$. II has at most (2^{2^κ}) possible partial plays against τ [the number of filters on κ]. Hence, I has at most (2^{2^κ}) possible answers (A_i, g_i) according to τ. If we denote by X the union of the images of all such possible g_i, then $|X| \leqq 2^{2^\kappa}$. Hence, for some $\delta < (2^{2^\kappa})^+$, we have an isomorphism $h : X \underset{\sim}{\to} \delta$. If we replace each g_i by $h \circ g_i$, we obtain from τ a winning strategy for I in $G^*(F, \delta)$, a contradiction.

3.7. Consequences. We summarize our remarks as follows. "For all $g \in On^\kappa$, II has a winning strategy in $G^*(F, g)$" iff "For all $g \in On^\kappa$, there exists some $\alpha \in On$ such that II has a winning strategy in $G(F, g, \alpha)$" iff "For all $\delta < (2^{2^\kappa})^+$, II has a winning strategy in $G^*(F, \delta)$".

DEFINITION 1. Let F be a filter over κ. F is *weakly precipitous* iff

(a) F is normal,

(b) II wins $G^*(F, (2^{2^\kappa})^+)$.

κ is weakly precipitous iff there exists a weakly precipitous filter over κ.

4. Semi-precipitous filters

4.1. We shall now consider a different property. Similar properties have been considered in [S] and [S-1].

DEFINITION 2. Let λ be a cardinal such that $\lambda > \kappa$. We say that κ is λ-semi-precipitous iff there exists a set of conditions P such that the following is forced over P: "there exists an elementary embedding $j: H_\lambda \to M$, of critical point κ, such that M is transitive".

Of course, H_λ means $(H_\lambda)^V$.

We shall now give a game-theoretic equivalence of λ-semi-precipitousness.

DEFINITION 3. Let R be a set. We say that R is *κ-plain* iff the following holds:

(a) $R \neq \emptyset$,

(b) R is a set of normal filters over κ,

(c) For all $F \in R$ and $A \in F^+$, $F[A] \in R$.

4.2. The game $H_R(F, \lambda)$. Let R be a κ-plain set and $g \in On^\kappa$. We define a game, $H_R(F, g)$, of length ω, as follows. Set $F_0 = F$. For $1 \leqq i < \omega$, player I will play as move number i a pair (A_i, g_i), where $A_i \subseteq \kappa$ and $g_i <_F g$, while player II will play a pair (F_i, γ_i), where $F_i \in R$ and $\gamma_i \in On$. The rules are as follows:

(a) For $0 \leqq i < \omega$, $A_{i+1} \in (F_i)^+$.

(b) For $0 \leqq i < \omega$, $F_{i+1} \supseteq F_i[A_{i+1}]$.

Assume that the game is over. Then, player II wins it iff for all i, k, n such that $1 \leqq i, k \leqq n < \omega$, $(g_i <_{F_n} g_k) \to (\gamma_i < \gamma_k)$.

For $\lambda \in On$, $H_R(F, \lambda)$ denotes the game $H_R(F, c_\lambda^\kappa)$, where $c_\lambda^\kappa: \kappa \to \{\lambda\}$.

If R is the set of all normal filters over κ, $H_R(F, g)$ will be denoted by $H(F, g)$.

THEOREM 4. *Let λ be a cardinal such that* $\text{cof}(\lambda) > \kappa$. *The following are equivalent*:

(1) *κ is λ-semi-precipitous.*

(2) *There exists a κ-plain set R such that, for all $F \in R$, player II has a winning strategy in the game $H_R(F, \lambda)$.*

(3) *There existts a normal filter F on κ such that II has a winning strategy in the game $H(F, \lambda)$.*

PROOF OF THEOREM 4. (2)→(3) is evident. Hence, we are going to prove (1)→(2) and (3)→(1).

(1)→(2). Assume that κ is λ-semi-precipitous. Let P be a set of conditions satisfying Definition 2, and let A denote the boolean completion of P. Let $M, j \in V^A$ be such that $\Vdash_A$ "$j: H_\lambda \to M$ is an elementary embedding of critical point κ and M is transitive". Let $D \in V^A$ be such that

$$\Vdash_A D = \{X \in H_\lambda \mid X \subseteq \kappa \text{ and } \kappa \in j(X)\}.$$

For all $p \in A - \{0\}$, set $F_p = \{X \subseteq \kappa \mid p \Vdash X \in D\}$. Since D is forced to be a V-normal, V-ultrafilter over κ, F_p is a normal filter over κ.

CLAIM 1. (a) $p \leqq q \to F_q \subseteq F_p$.
(b) $S \in (F_p)^+$ iff, for some $q \leqq p$, $q \Vdash S \in D$ iff, for some $q \leqq p$, $S \in F_q$.
(c) Assume that $S \in (F_p)^+$. Then: for some $q \leqq p$, $F_p[A] = F_q$.

PROOF OF CLAIM 1. (a) and (b) are obvious. Hence, let us prove (c).

Assume that $S \in (F_p)^+$. Set $q = \| S \in D \|^A \wedge p$. We claim that $F_p[S] = F_q$. By (b), $q > 0$. On the other hand, $F_p[S] \subseteq F_q$ is obvious. Hence, let us prove that $F_q \subseteq F_p[S]$. Assume not. Then, for some $X \in F_q$, $X \notin F_p[S]$. Hence, we can find some $Y \subseteq S$ such that $Y \in (F_p)^+$ but $Y \cap X = \emptyset$. Since $Y \in (F_p)^+$, we can find some $r \in A - \{0\}$ such that $r \leqq p$ and $r \Vdash Y \in D$. Since $Y \subseteq S$, $r \Vdash S \in D$. Hence, $r \leqq q$. Hence, $r \Vdash X \in D$, $Y \in D$, $X \cap Y = \emptyset$, a contradiction.

QED Claim 1

Hence, set $R = \{F_p \mid p \in A - \{0\}\}$. By Claim 1, R is κ-plain. We have got to show that, for all $F \in R$, II has a winning strategy in $H_R(F, \lambda)$. We shall define such a strategy, say σ, as follows. First, we fix some $p_0 \in A - \{0\}$ such that $F = F_{p_0}$. Assume that I plays, as first move in $H_R(F_{p_0}, \lambda)$, (A_1, g_1), where $A_1 \in (F_{p_0})^+$ and $g_1 : \kappa \to \lambda$. Let $p_1 \in P$ and $\gamma_1 \in On$ be such that $p_1 \leqq p_0$, $p_1 \Vdash A_1 \in D$ and $p_1 \Vdash j(g_1)(\kappa) = \gamma_1$. The answer of II to this move, according to σ, will be the pair (F_{p_1}, γ_1). [Hence, II has played, not only (F_1, γ_1), but (F_1, γ_1, p_1), where $F_1 = F_{p_1}$.]

Continuing in this way, we arrive, after n steps, at a move (F_{p_n}, γ_n, p_n) of player II. If I answers (A_{n+1}, g_{n+1}), we let II play by σ $(F_{p_{n+1}}, \gamma_{n+1})$, where (p_{n+1}, γ_{n+1}) are to (A_{n+1}, g_{n+1}, p_n) what (p_1, γ_1) is to (A_1, g_1, p_0). We claim that this strategy σ is winning. For, assume that the play is completed, that i, k, n are such that $1 \leqq i, k \leqq n < \omega$ and $g_i <_{F_n} g_k$. Set $X = \{\alpha < \kappa \mid g_i(\alpha) < g_k(\alpha)\}$. Hence,

$X \in F_n = F_{p_n}$. Hence, $p_n \Vdash \kappa \in j(X)$, hence $p_n \Vdash j(g_i)(\kappa) < j(g_k)(\kappa)$. On the other hand, $p_n \leqq p_i, p_k$. Hence, $p_n \Vdash \gamma_i < \gamma_k$. QED (1)→(2)

REMARK. Actually, the following stronger property is satisfied as well.

CLAIM 2. If $1 \leqq i, k \leqq n < \omega$, then $(g_i <_{F_n} g_k) \leftrightarrow (\gamma_i < \gamma_k)$.

PROOF OF CLAIM 2. Assume that $g_i \not<_{F_n} g_k$. Set $X = \{\alpha < \kappa \mid g_k(\gamma) \leqq g_i(\alpha)\}$. Hence, $X \in (F_n)^+$. Hence, for some $r \leqq p_n$, $r \Vdash X \in D$. As before, $r \Vdash \kappa \in j(X)$, hence, since $r \leqq p_i, p_k$, $r \Vdash \gamma_k \leqq \gamma_i$. QED Claim 2

(3)→(1). Let us assume that (3) is true, and let F be as in (3). Set $\theta = \lambda^\kappa$ and $P = \mathrm{Coll}(\omega, \theta)$ [P is hence the set of all finite partial functions $p : \omega \to \theta$, ordered by reverse inclusion]. We shall show that this set P satisfies Definition 2 for λ. Let σ be a fixed winning strategy for player II in the game $H(F, \lambda)$. Let G be a fixed P-generic set over V. Set $\boldsymbol{S} = (\mathscr{P}(\kappa) \times \lambda^\kappa)^V$. In $V[G]$, $\boldsymbol{S}$ is countable. Hence, we can find a surjection $h : \omega - \{0\} \to S$, such that, for all $s \in S$, $\{i \mid h(i) = s\}$ is infinite. Set, for $i \in \omega - \{0\}$, $h(i) = (A_i, g_i)$.

We are going to construct, in $V[G]$, a V-normal, V-ultrafilter D on κ, such that $((H_\lambda)^\kappa \cap V)/D$ is well-founded, by playing a certain play of the game $H(F, \lambda)$ against σ [of course, the full play will not be in V, but every initial segment of it will, being finite]. Hence, player II will always play according to σ, and we have to describe the moves of player I.

Assume that $1 \leqq n < \omega$ and that $n - 1$ moves have already been done. Say that I has played $((C_1, g_1), \ldots, (C_{n-1}, g_{n-1}))$, where

(a) $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_{n-1}$,

(b) $g_1, \ldots, g_{n-1}$ are as in the enumeration h of $\boldsymbol{S}$.

Assume that II has answered $((F_1, \gamma_1), \ldots, (F_{n-1}, \gamma_{n-1}))$. We have got to define (C_n, g_n). Let us first look at the pair (A_n, g_n) given by the enumeration h.

Case 1. $A_n \in (F_{n-1})^+$. Set $B_n = A_n \cap C_{n-1}$ (and $C_0 = \kappa$ if $n = 1$).

Case 2. $A_n \notin (F_{n-1})^+$. Set $B_n = C_{n-1} \cap (\kappa - A_n)$.

Hence, $B_n \in (F_{n-1})^+$.

Case a. For no $\alpha \in On$, $(g_n)^{-1}(\{\alpha\}) \in (F_{n-1}[B_n])^+$. Set $C_n = B_n$.

Case b. For some $\alpha \in On$, $(g_n)^{-1}(\{\alpha\}) \in (F_{n-1}[B_n])^+$. Let α_n be the least such α, and set $C_n = B_n \cap (g_n)^{-1}(\{\alpha_n\})$.

Hence, (C_n, g_n) is defined and $C_n \in (F_{n-1})^+$, $C_n \subseteq C_{n-1}$.

At the end of the ω moves, set $D = \bigcup \{F_n \mid n < \omega\}$ and

$$D' = \{X \in V \mid X \subseteq \kappa \text{ and for some } n \in \omega - \{0\},\ C_n \subseteq X\}.$$

CLAIM 3. $D = D'$ and D is a V-normal, V-ultrafilter on κ.

PROOF OF CLAIM 3. (a) Clearly, D is a filter on $\mathcal{P}(\kappa)^V$.

(b) $D' \subseteq D$. For, if $C_n \subseteq X \subseteq \kappa$ and $X \in V$, then $X \in F_n$, since $C_n \in F_n$.

(c) Assume that $X \in \mathcal{P}(\kappa)^V$. Then: $X \in D'$ or $(\kappa - X) \in D'$. To see this, take some n such that $1 \leqq n < \omega$ and $X = A_n$. Hence, $C_n \subseteq X$ or $C_n \subseteq \kappa - X$.

(a), (b), (c) together show that $D = D'$ and D is an ultrafilter on $\mathcal{P}(\kappa)^V$.

(d) D is V-normal. Since $F \subseteq D$, D is uniform. Hence, let $f \in V$ such that $f: \kappa \to \kappa$ and f is regressive. We have got to find some $X \in D$ such that $f \restriction_X$ is constant. Let $n < \omega$ be such that $f = g_n$. Since $F_{n-1}[B_n]$ is normal and g_n is regressive, for some $\alpha < \kappa$, $(g_n)^{-1}(\{\alpha\}) \in F_{n-1}[B_n]^+$. Hence, we are in Case b. Hence $f \restriction_{C_n}$ is constant and $C_n \in D$. QED Claim 3

Now, we can form the Ultrapower $(H_\lambda)^\kappa \cap V/D$ [which is, in this case, equal to $((H_\lambda)^\kappa \cap H_\lambda)/D$, since $\mathrm{cof}(\lambda) > \kappa$]. The Theorem of Los is clearly true, since we take all functions $f: \kappa \to H_\lambda$ such that $f \in V$. Let this ultrapower be denoted by M, and let $j: H_\lambda \to M$ be the canonical elementary embedding. Since D is V-normal and uniform, j is of critical point κ. Hence, it is enough to show the following.

CLAIM 4. *M is well-founded.*

PROOF OF CLAIM 4. Assume not. Let $(f_n)_{n<\omega} \in V[G]$ be a sequence of functions such that, for all $n < \omega$, $f_n \in (H_\lambda)^\kappa \cap V$ and

$$\{\alpha < \kappa \mid f_{n+1}(\alpha) \in f_n(\alpha)\} \in D.$$

Note that $H_\lambda \subseteq V_\lambda$ [this is obvious for regular λ, and is hence true for limit λ too, by taking unions]. Hence, replacing if necessary each f_n by the function $\mathrm{rank} \circ f_n$, we can assume that, actually, for $n < \omega$, $f_n \in \lambda^\kappa \cap V$. For $n < \omega$, let $i(n)$ be the least index i such that $1 \leqq i < \omega$ and $f_n = g_{i(n)}$. We claim that $\gamma_{i(n+1)} < \gamma_{i(n)}$ [where γ_i is part of the answer of player II at the move number i, according to σ]. To see that, let $k < \omega$ be such that $i(n)$, $i(n+1) \leqq k$ and $\{\alpha < \kappa \mid f_{n+1}(\alpha) < f_n(\alpha)\} \in F_k$. By the rules of the game $H(F, \lambda)$ we see that $\gamma_{i(n+1)} < \gamma_{i(n)}$. QED Claim 4. Theorem 4

4.3. REMARKS.

(1) If we replaced, in Definition 2, H_λ by V_λ, then Theorem 4 would be true for every *ordinal* λ such that $\mathrm{cof}(\lambda) > \kappa$.

(2) The clause $\mathrm{cof}(\lambda) > \kappa$ is not used in the proof of (3)$\to$(1), since we can take as model the set $M = ((H_\lambda)^\kappa \cap V)/D$ [or $((V_\lambda)^\kappa \cap V)/D$]. It is used in the proof of (1)$\to$(2) to show that, if $g \in \lambda^\kappa \cap V$, then $g \in H_\lambda$ [or $g \in V_\lambda$], so that $j(g)$ be defined.

(3) Let us keep the notations of the proof of (3)→(1), where we can assume that M is transitive, since it is well-founded. Set $\delta = On \cap M$. Since $On \cap M = (\lambda^\kappa \cap V)/D$ and since $V[G] \models |(\lambda^\kappa)^V| = \omega$, we see that $\delta < (\omega_1)^{V[G]}$. Hence, setting $\mu = [(\lambda^\kappa)^+]^V$, we see that $\delta < \mu$. But, if we look now at the proof of (1)→(2), we see that, for $F \in R$, II has a winning strategy in the game $H_R(F, \lambda)$, while playing pairs (F_i, γ_i) with $\gamma_i < \delta$.

Hence, for $\mathrm{cof}(\lambda) > \kappa$, if II wins $H_R(F, \lambda)$, then, for some ordinal $\delta < (\lambda^\kappa)^+$, II has already a winning strategy in the game with the additional requirement that all ordinals γ_i played have to be $< \delta$.

DEFINITION 5. κ is semi-precipitous iff, for all $\lambda > \kappa$, κ is λ-semi-precipitous.

4.4. REMARKS.

(1) It is clear that if κ is λ-semi-precipitous and $\mu \leqq \lambda$, then κ is μ-semi-precipitous.

(2) Set $\lambda = (2^{2^\kappa})^+$ and assume that κ is λ-semi-precipitous. Then: κ is weakly precipitous. To see this, let F be a normal filter on κ such that F satisfies condition (3) of Theorem 4. It is enough to show that F is weakly precipitous. Let σ be a winning strategy for player II in $H(F, \lambda)$. Then, clearly, the same σ is a winning strategy for player II in the game $G(F, \lambda, On)$, and we conclude by Remark 3.6(3).

We shall show later that weakly precipitous does not imply semi-precipitous.

(3) Assume that κ is semi-precipitous. For all regular $\lambda > \kappa$, let R_λ be a κ-plain set satisfying condition (2) of Theorem 4. We can find a cofinal class $S \subseteq On$ and a κ-plain set R such that, for all $\lambda \in S$, $R_\lambda = R$. Hence clearly, for all $\lambda \in On$ and all $F \in R$, player II has a winning strategy in the game $H_R(F, \lambda)$. Hence, the following are equivalent:

(a) κ is semi-precipitous.

(b) For some κ-plain R, for all $F \in R$, II has a winning strategy in $H_R(F, \lambda)$ for all $\lambda \in On$.

(c) For some normal filter F on κ, II has a winning strategy in $H(F, \lambda)$, for all $\lambda \in On$.

5. Consistency results

THEOREM 6. *Assume that $O^\#$ exists. Then: every Silver indiscernible is semi-precipitous in L.*

Proof of Theorem 6. Let $\{c_i \mid i \in On\}$ denote the canonical indiscernibles for L. We shall show the following.

Claim 1. $L \models$ "c_0 is c_1-semi-precipitous".

Let us show that Claim 1 implies the Theorem. For, we then have that, for all α such that $c_0 < \alpha < c_1$, $L \models$ "c_0 is α-semi-precipitous". Since $L_{c_1} \prec L$, we see that $L_{c_1} \models$ "c_0 is semi-precipitous". Again because $L_{c_1} \prec L$, we get that $L \models$ "c_0 is semi-precipitous".

Proof of Claim 1. Let $\pi : L \to L$ be an elementary embedding, such that $\pi \restriction_{c_0} = \mathrm{Id} \restriction_{c_0}$, $\pi(c_0) = c_1$ and $\pi(c_1) = c_2$. Set $j = \pi \restriction_{L_{c_1}}$. Hence, j is an elementary embedding from L_{c_1} into L_{c_2}, of critical point κ, and $j \in V$.

Set $P = \mathrm{Coll}(\omega, c_1) = \mathrm{Coll}(\omega, c_1)^L$. Since $\mathscr{P}(P)^L$ is countable in V, there exists $G \in V$ such that G is P-generic over L.

Claim 2. $L[G] \models$ "there exists an elementary embedding $j' : L_{c_1} \to L_{c_2}$ of critical point c_0".

Proof of Claim 2. Let us, for a moment, work in $L[G]$. For $i = 1, 2$, set

$$X_i = \{(\varphi, a) \mid \varphi \text{ is a formula and } a \in [L_{c_i}]^{<\omega} \text{ and } L_{c_i} \models \varphi[a]\}.$$

Since $|c_1| = \omega$, we can find a family $(A_i)_{i<\omega}$ such that $A_i \subseteq A_{i+1}$, $|A_i| < \omega$ and $L_{c_1} = \bigcup \{A_i \mid i < \omega\}$.

Let T be the set of all partial functions p such that, for some $i < \omega$, the following holds:

(i) $\mathrm{Dom}(p) = A_i$ and $\mathrm{Rg}(p) \subseteq L_{c_2}$.

(ii) If $(\varphi, a) \in X_1$ and $a \in [A_i]^{<\omega}$, then $(\varphi, p(a)) \in X_2$.

(iii) If $\alpha < c_0$ and $\alpha \in A_i$, then $p(\alpha) = \alpha$.

(iv) If $c_0 \in A_i$, then $p(c_0) > c_0$.

If $p, q \in T$, we set $p \lhd q$ iff $p \subseteq q$ and $p \neq q$.

The existence of j shows that $(T, \lhd)$ is not well-founded in V. Hence, it is not well-founded in $L[G]$. If b is a cofinal branch through T, it is clear that $j' = \bigcup b$ is an elementary embedding from L_{c_1} to L_{c_2}, of critical point κ. QED Theorem 6

The same kind of argument will give us the following.

Theorem 7. *Assume that λ is a cardinal in V, such that $V \models$ "κ is λ-semi-precipitous and* $\mathrm{cof}(\lambda) > \kappa$". *Then $L \models$ "κ is λ-semi-precipitous".*

Proof of Theorem 7. Set $\theta = (\lambda^{\kappa})^V$ and $P = \mathrm{Coll}(\omega, \theta) = \mathrm{Coll}(\omega, \theta)^L$. Let G be P-generic over V. The proof of Theorem 4 shows the existence in

$V[G]$ of an elementary embedding $J: L_\lambda \to M$ of critical point κ, where M is transitive. It is clear that for some $\mu \in On$, $M = L_\mu$. Since $L_\lambda, L_\mu \in L[G]$, and $L[G] \models$ "$|\lambda| = \omega$", we can apply the tree argument of the proof of Theorem 6 between $L[G]$ and $V[G]$ and find an elementary embedding $j': L_\lambda \to L_\mu$ of critical point κ, such that $j' \in L[G]$. Since $P \in L$, we conclude that κ is λ-semi-precipitous in L. QED Theorem 7

5.1. Remarks. Hence, from the existence of $0^\#$, we obtain a semi-precipitous cardinal in L. However, it is easily shown that, if $V = L$, then a semi-precipitous cardinal is strongly inaccessible. Indeed, it is easily shown that, if $V = L$ and κ is κ^+-semi-precipitous, then κ is completely ineffable. Hence, in order to obtain a semi-precipitous filter on ω_1, we have to do a further forcing extension [where we say that a normal filter F on κ is semi-precipitous iff, for all $\lambda \in On$, player II wins $H(F, \lambda)$]. Hence:

Theorem 8. *Assume that κ is a regular cardinal, and that P is a set of conditions such that P has the κ-antichain condition. Assume that λ is a regular cardinal such that $\lambda > \kappa$ and $\lambda > |P|$. Assume, moreover, that $V \models$ "κ is λ-semi-precipitous". Then, $V^P \models$ "κ is λ-semi-precipitous".*

Proof of Theorem 8. Assume not. Then, without loss of generality, $\Vdash_P$ "κ is not λ-semi-precipitous". Let, in V, $\theta \geqq \lambda^\kappa$ and $Q = \mathrm{Coll}(\omega, \theta)$. Let K be Q-generic over V. By assumption, there exists in $V[K]$ an elementary embedding $j: H_\lambda \to M$ of critical point κ, where M is transitive. Without loss of generality, $V[K] \models$ "$|M| = \omega$" [if not, we could increase θ in order to insure this conclusion]. Hence, there exists a set $G^* \in V[K]$, such that G^* is $j(P)$-generic over M [it is clear that we can assume that $P \in H_\lambda$, and that $\mathscr{P}(P) \subseteq H_\lambda$]. Since P has the κ-antichain condition in V and since $j \restriction_\kappa = \mathrm{Id} \restriction_\kappa$, $j \restriction_P : P \to j(P)$ is V-complete. Hence, setting $G = j^{-1}(G^*)$, we see that G is P-generic over H_λ, hence over V. j clearly gives rise to an elementary embedding $j^*: H_\lambda[G] \to M[G^*]$, which is in $V[K]$. Since $|P| < \lambda$ in V, we see that $H_\lambda[G] = (H_\lambda)^{V[G]}$. Hence, in order to finish the proof, it is enough to show that

Claim 1. $V[K]$ is a generic extension of $V[G]$.

Proof of Claim 1. Let $\dot{G} \in V^{B(Q)}$ be a term which realizes with probability one the P-generic set G over V which has just been constructed. Let us define, in V, a map $h: B(P) \to B(Q)$ setting, for $a \in B(P)$, $h(a) = \| a \in \dot{G} \|^{B(Q)}$. Since $\dot{G}$ is forced over Q to be generic over V, h is a complete boolean morphism (and, for every Q-generic set K over V, $h^{-1}(K) = (\dot{G})_K$). It is, however, possible that h be not injective. Hence, set

$$b = \inf_{B(P)} \{a \in B(P) \mid h(a) = 1\} \quad \text{and} \quad B' = \{a \in B(P) \mid a \leqq b\}.$$

Since h is complete, $h(b) = 1$. Moreover, it is clear that $h \restriction_{B'}$ is injective. Now, let K be Q-generic over V and set $G = (\dot{G})_K$. Since $h(b) = 1$, we see that $b \in G$. Hence, $G' = G \cap B'$ is B'-generic over V and $V[G] = V[G']$. But, since $h' = h \restriction_{B'}$ is injective and since $(h')^{-1}(K) = G'$, we see that $V[K]$ is a generic extension of $V[G']$, hence of $V[G]$. QED Claim 1, Theorem 8

5.2. *Consequences*. Hence, from the existence of $0^{\#}$, we have deduced the existence of a set-generic extension of L, in which ω_1 is semi-precipitous, hence weakly precipitous. Hence, the existence of a weakly precipitous filter (even of a semi-precipitous one) on ω_1 does not imply the existence of $0^{\#}$. Of course we could, by Theorem 7, replace in this remark "$0^{\#}$ exists" by "there exists a semi-precipitous cardinal".

5.3. *Complements*. We will now show that the existence of a cardinal κ which is $(2^{2^\kappa})^{++}$-semi-precipitous is, consistencywise, strictly stronger that the existence of a κ which is $(2^{2^\kappa})^{+}$-semi-precipitous. In particular, the existence of a semi-precipitous cardinal is, consistencywise, strictly stronger that the existence of a weakly precipitous cardinal.

THEOREM 9. *Assume that $V = L$ and that n is an integer such that $2 \leqq n$. Assume that κ is κ^{+n+1}-semi-precipitous. then: there exists an $\alpha < \kappa$ such that α is α^{+n}-semi-precipitous.*

5.4. *Comments*. Assume hence that, in V, κ is $(2^{2^\kappa})^{++}$-semi-precipitous. Using Theorem 7, we see that $L \models$ "κ is κ^{+4}-semi-precipitous". Using Theorem 13, we can find some $\alpha < \kappa$ such that $L \models$ "α is α^{+3}-semi-precipitous". Since κ is strongly inaccessible in L, $L_\kappa \models$ "ZFC + α is α^{+3}-semi-precipitous". In particular, $L_\kappa \models$ "α is weakly precipitous".

5.5. PROOF OF THEOREM 9. Assume that κ is κ^{+n+1}-semi-precipitous. Set $P = \mathrm{Coll}(\omega, \kappa^{+n+1})$. Let G be P-generic over V. By Theorem 4, we can find some $\mu \in On$ and, in $V[G]$ ($= L[G]$), an elementary embedding $j : L_{(\kappa^{+n+1})} \to L_\mu$, of critical point κ.

Since κ is κ^{+n}-semi-precipitous, we can find, by Theorem 4 and Remark 4.3(3), a normal filter F on κ and an ordinal $\delta < \kappa^{+n+1}$ such that player II has a winning strategy in the game $H(F, \kappa^{+n})$ with the additional requirement that he must play only ordinals $\gamma_i < \delta$. Let σ denote such a strategy. Due to the bound δ we see that, since $n \geqq 2$, $\sigma \in L_{(\kappa^{+n+1})}$. In particular, $\sigma \in L_\mu$. Hence, $L_\mu \models$ "σ is a winning strategy for player II in the game $H(F, \kappa^{+n})$". By elementarity of j, for some $\alpha < \kappa$, $L_{(\kappa^{+n+1})} \models$ "for some normal filter F' on α and some σ', σ' is a winning strategy for player II in the game $H(F', \alpha^{+n})$". It is clear that the pair (F', σ') satisfies the same statement in L. QED Theorem 9

5.6. Remarks.

(1) Hence, winning $G^*(F, g)$ is a local property, i.e.: if player II wins $G^*(F, (2^{2^\kappa})^+)$ then, for all $\lambda \in On$, he wins $G^*(F, \lambda)$.

(2) On the contrary, winning $H(F, \lambda)$ is not a local property. Player II may win $H(F, (2^{2^\kappa})^+)$ without winning $H(F, (2^{2^\kappa})^{++})$.

6. More on weakly precipitous filters

We have shown that semi-precipitous filters may exist in L, that the existence of a semi-precipitous filter is already a "medium large cardinal" axiom, and that a semi-precipitous filter on ω_1 may exist in a set-generic extension of L. Since we know much less about weakly precipitous filters, the following theorem is not irrelevant.

Theorem 10. *Assume that κ is a regular cardinal, that F is a weakly precipitous filter on κ and that P is a set of conditions which satisfies the κ-antichain condition. Then, $V^P \models$ "the filter generated by F on κ is weakly precipitous".*

Before proving Theorem 10, we shall have to recall a few constructions and lemmas.

6.1. Let X be a set. Let P be a set of conditions, and let us denote by A the boolean completion of P. To each $a \in V^A$ such that $\Vdash_A a \subseteq X$, we can associate a function $h_a \in A^X$ defined by the formula $h_a(x) = \| x \in a \|^A$ (for $x \in X$).

It is clear that $\Vdash_A a = (h_a)^{-1}(G)$, where G denotes the A-generic. Hence, the correspondence between a and h_a is a bijection from $\{a \in V^A \mid \Vdash_A a \subseteq X\}$ onto A^X. Moreover, this bijection is boolean, in the following sense:

if $a, b, c \in V^A$, then $\Vdash_A a \cap b = c$ iff $h_a \wedge h_b = h_c$, where, for $x \in X$, we set $(h_a \wedge h_b)(x) = h_a(x) \wedge h_b(x)$ in A.

6.2. Now let, in addition, F be a filter on X. For $f, g \in A^X$, set $f \leqq_F g$ iff $\{x \in X \mid f(x) \leqq g(x)\} \in F$ and $f =_F g$ iff $\{x \in X \mid f(x) = g(x)\} \in F$.

Set $A^X/F = (A^X/=_F)$ (the reduced product). For $f \in A^X$, let $[f]_F$ denote its class in A^X/F and set $[f]_F \leqq [g]_F$ iff $f \leqq_F g$.

Hence, A^X/F is a boolean algebra, and we have a boolean morphism $j_F : A \to A^X/F$ given by $j_F(a) = [c_a^X]_F$, where $c_a^X : X \to \{a\}$ is the constant function.

Moreover, it is a well-known fact that j_F is complete iff, for some cardinal κ, A has the κ-antichain condition and F is κ-complete.

6.3. Now, let G be A-generic over V. In $V[G]$, set

$$\tilde{F}_G = \{S \subseteq X \mid \exists E \in F, E \subseteq S\}$$

(the filter over κ generated by F). Hence, $\tilde{F}$ will be the term of V^A representing $\tilde{F}_G$. By 6.1,

$$\mathscr{P}(X) \cap V[G] = \{h^{-1}(G)/h \in A^X \cap V\}.$$

Lemma 11. *For $h \in A^X \cap V$, $h^{-1}(G) \in \tilde{F}_G$ iff, for some $p \in G$, $j_F(p) \leqq [h]_F$.*

Proof of Lemma 11. Assume that $h^{-1}(G) \in \tilde{F}_G$. Then, for some $E \in F$, $E \subseteq h^{-1}(G)$. Take $p \in G$ such that $p \Vdash_A B \subseteq h^{-1}(G)$. Then, for $x \in E$, $p \Vdash_A h(x) \in G$, i.e. $p \leqq h(x)$. Since $E \in F$, we see that $j_F(p) \leqq [h]_F$.. These derivations are actually easily seen to be equivalences. QED Lemma 11

Lemma 12. *Assume that $j_F : A \to A^X/F$ is complete. Then, for all $h \in A^X \cap V$ and all $p \in A - \{0\}$, $p \Vdash_A h^{-1}(G) \in \tilde{F}$ iff $j_F(p) \leqq [h]_F$.*

Proof of Lemma 12. The "if" direction follows from Lemma 11. Let us hence prove the "only if" direction. Assume that $p \Vdash_A h^{-1}(G) \in \tilde{F}$. Set $D = \{q \leqq p \mid j_F(q) \leqq [h]_F\}$. By Lemma 11, D is dense beneath p in A. Since j_F is complete, we see that $j_F(p) = \sup(\{j_F(q)/q \in D\}) \leqq [h]_F$. QED Lemma 12

6.4. Let us still assume that $j_F : A \to A^X/F$ is complete. Let $\pi_F : A^X/F \to A$ denote the normal projection associated to the complete embedding j_F.

Lemma 13. *Let $h \in A^X \cap V$. Then: $\| h^{-1}(G) \in (\tilde{F})^+ \|^A = \pi_F([h]_F)$. Hence, $h^{-1}(G) \in (\tilde{F}_G)^+$ iff $\pi_F([h]_F) \in G$.*

Proof of Lemma 13. Define a function $g \in A^X \cap V$ setting, for $x \in X$, $g(x) = 1 - f(x)$ (in A). Take $p \in A$. By Lemma 12, we see that $p \Vdash h^{-1}(G) \in (\tilde{F})^+$ iff $p \Vdash g^{-1}(G) \notin \tilde{F}$ iff $(\forall q \leqq p)[q \nVdash g^1(G) \in \tilde{F}]$ iff $(\forall q \leqq p)[j_F(q) \nleqq [g]_F]$ iff $(\forall q \leqq p)[j_F(q) \wedge [h]_F > 0]$.

Hence, summarizing, $p \Vdash h^{-1}(G) \in (\tilde{F})^+$ iff $(\forall q \leqq p)[j_F(q) \wedge [h]_F > 0]$.

In order to conclude the proof of Lemma 13, we hence need only the following:

CLAIM 1. Assume that $j: A \to B$ is a complete morphism between boolean algebras, admitting an associated normal projection $\pi: B \to A$. Take $p \in A$ and $b \in B$. Then: $(\forall q \leqq p)[j(q) \wedge b > 0]$ iff $p \leqq \pi(b)$.

PROOF OF CLAIM 1. Assume that $(\forall q \leqq p)[j(q) \wedge b > 0]$, but $p \nleqq \pi(b)$. For some q, $0 < q \leqq p$ and $q \wedge \pi(b) = 0$. Hence, $j(q) \wedge b = 0$, for, if not, it would project on an element $r \in A$ such that $r \leqq q$, $\pi(b)$. The converse is clear.

QED Lemma 13

6.5. Proof of Theorem 10. Hence we let F be, in V, a weakly precipitous filter on some fixed regular cardinal κ. Let A denote the boolean completion of P, and let G be A-generic over V. We have got to show that, in $V[G]$, $\tilde{F}_G$ is a weakly precipitous filter on κ.

Let θ be an ordinal such that $V[G] \models$ "θ is a cardinal and $\theta \geqq (2^{2^\kappa})^+$". By 3.7, it is enough to show that player II has a winning strategy in the game $G^*(\tilde{F}_G, \theta)$. We shall now describe such a strategy.

By 3.7 there exists an ordinal δ such that, in V, player II has a winning strategy in the game $G(F, \theta, \delta)$. Let σ b a fixed winning strategy for player II in this game (in V). We shall actually construct a winning strategy (in $V[G]$), for player II in the game $G(\tilde{F}_G, \theta, \delta)$, say $\tilde{\sigma}$. Assume that, in $V[G]$, I plays, in $G(\tilde{F}_G, \theta, \delta)$, a pair (X_1, f_1), where $X_1 \in (\tilde{F}_G)^+$ and $f_1: \kappa \to \theta$. Set $F_0 = F$.

Let $h_1 \in A^\kappa \cap V$ be such that $X_1 = (h_1)^{-1}(G)$ and, applying the maximum principle in V, let $w_1 \in (V^A)^\kappa$ be such that

(a) for all $\alpha < \kappa$, $(w_1(\alpha))_G = f_1(\alpha)$,

(b) for all $\alpha < \kappa$, $\Vdash_A$ "$w_1(\alpha) \in On$ and $w_1(\alpha) < \check{\theta}$".

CLAIM 1. There exists, in V, a function $h_1^* \in A^\kappa$, a function $g_1 \in \theta^\kappa$, a normal filter F_1 over κ and an ordinal $\alpha_1 < \delta$ such that:

(1) for all $\alpha < \kappa$, $h_1^*(\alpha) \leqq h_1(\alpha)$,

(2) for all $\alpha < \kappa$, if $h_1^*(\alpha) > 0$, then $h_1^*(\alpha) \Vdash_A w_1(\alpha) = (g_1(\alpha))^\vee$,

(3) $(h_1^*)^{-1}(G) \in ((\tilde{F}_1)_G)^+$,

(4) if we set $A_1 = \{\alpha < \kappa \mid h_1^*(\alpha) > 0\}$, then $(F_1, \alpha_1) = \sigma(A_1, g_1)$.

REMARK. Since $(h_1^*)^{-1}(G) \subseteq A_1$, point (3) shows that $A_1 \in (F_1)^+$; hence, $\sigma(A_1, g_1)$ is defined.

PROOF OF CLAIM 1. As in the previous sections, we define A^κ/F and

$j_F : A \to A^\kappa/F$. Since F is κ-complete and A has the κ-antichain condition, j_F is complete. Hence, we shall denote by π_F its associated normal projection.

Set $p_1 = \pi_F([h_1]_F)$. By Lemma 13, $p_1 \in G$. Hence, in order to prove Claim 1, it is enough to show that the following set:

$$D = \{ p \in A \mid \text{there exists } h_1^*, g_1, F_1, \alpha_1 \text{ satisfying points (1), (2), (4) of Claim 1 and such that } \pi_{F_1}([h_1^*]_{F_1}) \geqq p \}$$

is dense in A beneath p_1.

For, if this is the case, since G is generic and $p_1 \in G$, we can find $h_1^*, g_1, F_1, \alpha_1$ satisfying points (1), (2), (4) of Claim 1, such that $\pi_{F_1}([h_1^*]_{F_1}) \in G$. By Lemma 13, this implies that these four objects satisfy point (3) of Claim 1 as well.

To show the density of D beneath p_1, take $q \in A$, such that $0 < q \leqq p_1$. Hence, $j_F(q) \wedge [h_1]_F > 0$. Set $A_1 = \{\alpha < \kappa \mid q \wedge h_1(\alpha) > 0\}$. Hence, $A_1 \in F^+$. Define $g_1 \in \theta^\kappa$ and $h_1^* \in A^\kappa$ as follows:

(a) if $\alpha \notin A_1$, $h_1^*(\alpha) = 0$ (in A) and $g_1(\alpha) = 0$ (in On),

(b) if $\alpha \in A_1$, then $h_1^*(\alpha)$ and $g_1(\alpha)$ are chosen in such a way that

$$0 < h_1^*(\alpha) \leqq q \wedge h_1(\alpha) \quad \text{and} \quad h_1^*(\alpha) \Vdash_A (g_1(\alpha))^\vee = w_1(\alpha).$$

It is clear that conditions (1) and (2) of the claim are satisfied. Since $A_1 \in F^+$, we can set $(F_1, \alpha_1) = \sigma(A_1, g_1)$. Hence, condition (4) is satisfied too. Finally, let us set $p = \pi_{F_1}([h_1^*]_{F_1})$. In order to finish the proof of Claim 1 we only need to show that $0 < p \leqq q$. Since $A_1 \in F_1$, $[h_1^*]_{F_1} > 0$, hence $p > 0$. On the other hand, for $\alpha \in A_1$, $h_1^*(\alpha) \leqq q$, and hence $[h_1^*]_{F_1} \leqq j_{F_1}(q)$, which implies that $p \leqq \pi_{F_1}(j_{F_1}(q)) = q$. QED Claim 1

Back to the Proof of the Theorem. Hence, let us choose a quadruple $(F_1, g_1, h_1^*, \alpha_1)$ satisfying Claim 1. The answer of player II by $\tilde{\sigma}$ to (X_1, f_1) in $G(\tilde{F}_G, \theta, \delta)$ will be

$$\tilde{\sigma}(X_1, f_1) = (\tilde{F}_1[(h_1^*)^{-1}(G)], \alpha_1) = (H_1, \alpha_1),$$

say. This has a sense, since $(h_1^*)^{-1}(G) \in (\tilde{F}_1)^+$.

We shall now show how to construct the answer by $\tilde{\sigma}$ to the second move (X_2, f_2) of player I in $G(\tilde{F}, \theta, \delta)$.

Note that, since the G-game is an open game, we can always assume that σ is a positional strategy, i.e., that the value of σ depends only on the last move of player I.

Without loss of generality, we can assume that $X_2 \subseteq (h_1^*)^{-1}(G)$ and that, for all $\alpha \in X_2$, $f_2(\alpha) < f_1(\alpha)$. Let, as before, $w_2 \in (V^A)^\kappa \cap V$ and $h_2 \in A^\kappa \cap V$ be such that

(a) for all $\alpha < \kappa$, $(w_2(\alpha))_G = f_2(\alpha)$,

(b) $X_2 = (h_2)^{-1}(G)$,

(c) for all $\alpha < \kappa$, $h_2(\alpha) \leqq h_1^*(\alpha)$,

(d) for all $\alpha < \kappa$, $\Vdash_A$ "if $\alpha \in (h_2)^{-1}(G)$, then $w_2(\alpha) < w_1(\alpha)$".

Using a density argument identical to that of the proof of Claim 1, we can find a quadruple $(F_2, g_2, h_2^*, \alpha_2)$ satisfying Claim 1 with 1 added to every index. Hence, $\tilde{\sigma}(X_2, f_2)$ will be defined as $(\tilde{F}_2[(h_2^*)^{-1}(G)], \alpha_2)$. It is clear that we can continue in this way, and that the strategy $\tilde{\sigma}$ just described is winning for player II in $G(\tilde{F}, \theta, \delta)$. QED Theorem 10

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